I3T: Using Interactive Computer Graphics to Teach Geometric Transformations

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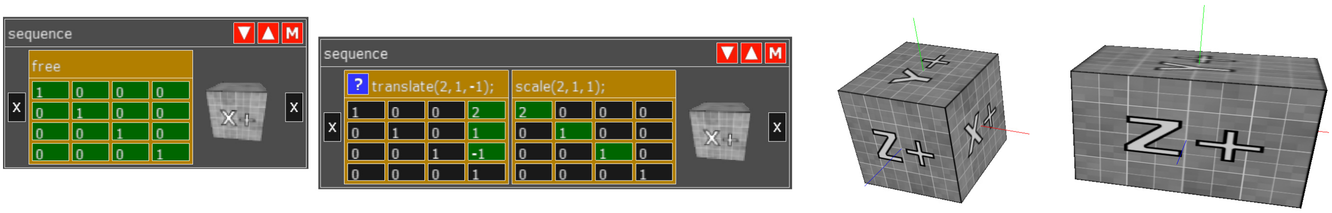

Figure 1: *The user of our I3T framework can interactively modify the transformation matrices (left) and immediately see the result in a 3D scene preview (right). The left sequence shows the application of identity matrix and the right sequence is an application of translation and scale. The right-hand side shows the effect of these transformations on an user-defined object.*

Abstract

Geometric transformations play an important role in a vast variety of disciplines. Although they belong to the fundamental concepts, they are also difficult to comprehend. Thousands of students take courses of algebra every year and although they may conceptually understand the transformations and mechanically solve the presented problems, they often struggle in visualizing the effect of the transformation on 3D objects represented as matrices. We explored the hypothesis that using interactive 3D computer graphics to visualize the transformations has its learning benefit. We have developed a novel framework for interactive 3D transformations called Interactive 3D Transformations (I3T) that allows for exploring and visualizing immediate effect of 3D transformations on rigid objects. We tested nine graduate students with I3T and compared them with the control group of another nine participants that used traditional passive methods. Moreover, we have tested the students spatial abilities by using a standardized test and we have evaluated how this affects their ability to comprehend the 3D transformations. Overall results showed that students increased their understanding of transformations between the pretest and posttest in both groups. When comparing the two groups, although the mean score in the posttest was two times higher for the I3T group, it did not show that this was statistically significantly higher than for the Traditional Group. The written responses showed higher enthusiasm of the students who used the interactive tool as opposed to using the passive learning method.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling— I.3.6 [Computer Graphics]: Methodology and Techniques—Interaction Techniques;

1. Introduction

Geometric transformations are among the most difficult and among the most important concepts in many engineering and scientific disciplines, such as architecture, physics, civil engineering, aerospace engineering, and manufacturing, to mention a few. Students are exposed to these concepts early in algebra and the most common way to explain them is by using matrix operations. A vertex is transformed by multiplying its coordinates by the corresponding 4×4 matrix and an object transformation is performed by transforming all its vertices. The 3D transformations can be expressed as matrix modifications; for example, modifying the main diagonal of the matrix corresponds to scaling and the last column values translate the vertex (see Figure 3). Moreover, transformation composition has its mathematical representation, as matrix multiplication and inverse operations are represented as inverse matrices.

The mathematical representation is straightforward, easy to comprehend, and students are usually able to solve problems by applying the matrix operations. However, bringing the transformation into a practical usage by *applying* them and understanding the effect of them, is usually hard. As documented in literature [Mat99], mental manipulation of 3D objects is a very difficult and often neglected task. While different levels of spatial thinking have been documented and measured [BG97, Bra00], the relationship of spatial thinking to the ability to understand mathematical representations to actually visualizing and applying them is a difficult and not well-studied problem. Moreover, students' ability to think spatially has been identified as a predictor of success in careers in sciences, technology, engineering and mathematics [New10, UMT*13]. Therefore, educators are posed with the challenge of supporting student reasoning through the use of multiple representations [BWM*09] as well as with equipping students with skills for thinking spatially [L*07]. Thousands of students are exposed to courses of algebra and, in particular, to the topics of mathematical representation of transformations every year. The topic itself is difficult and causes many problems and frustrations.

While traditional disciplines have been predominantly using textbooks and equations to teach transformations, Computer Graphics has been using geometric transformations from its very beginning [Hug14] and it is a well-studied problem, in particular in the context of animation [Par12]. Several Computer Graphics interactive applications have been developed and used to explain camera behavior or to demonstrate the capabilities of rendering engines such as OpenGL in [Rob01]. However, to the best of our knowledge, testing of the effectiveness of using interactive applications to teach transformations has been very limited.

In this paper, we introduce **I3T** (Interactive 3D Transformations), a novel framework for manipulating geometric transformations. Our tool (see Figure 1 and the accompanying video) allows the user to enter the matrix values, concatenate them, invert them, lock several elements (e.g., the diagonal to show scaling or the last column to show uniform translation), and modify their values. At the same time, an object (or a set of objects) is (are) displayed and the corresponding operation is animated on the screen.

We tested I3T on a group of mostly graduate students enrolled in an Introduction to Computer Graphics class. All 18 students were tested during the pretest and the spatial ability test [Gua80] was applied to them. Then, half of the participants was exposed to the traditional treatment and the other half used the I3T. The students who underwent traditional treatment had description in terms of equations and text, while the I3T group had a description of the usage of the tool. The students then performed the same tasks utilizing their corresponding instructional approach. After that, the posttest evaluated their learning gains. Our results show that that students increased their understanding of transformations between the pretest and posttest in both groups. When comparing the two groups, although the mean score in the posttest was two times higher for the I3T group, the t-test did not show a large, statistically significant rise in score for the Traditional Group. Moreover, the spatial abilities did not show any effect on the students learning. The written responses show much higher enthusiasm for using the interactive tool as opposed to using the passive learning method.

2. Related Work

We review related work in geometric transformations for learning and we categorize it into: (i) fundamentals of geometry for preschool to middle school students, (ii) kinematics for robotics, and (iii) Computer Graphics. We also briefly discuss the previous work on spatial thinking. We do not review the vast body of work of mathematics education in general and we refer readers to a survey on the topic of technology and mathematics education [Fey89] and to a recent survey on history of mathematics and education [Kil14].

K-12 Mathematics Education: Teaching geometry and geometric transformations fundamentals for preschool to middle school students has been initially performed without using technology. One of the first examples of using computers was by employing specialized versions of Logo turtle concept and Supposer program [SY86]. Later in the 80's, specialized interactive geometry programs entitled Dynamic Geometry Systems (DGS) such as Geometer's Sketchpad [Jac95], GeoGebra [Geoa], and GEONExT [GEOb] led to a higher geometrical understanding as documented by [Cle03]. An in-depth study was presented by Hollebrands *et al.* [Hol03] and it concluded that technology should be incorporated in the teaching and learning of mathematics, but "very little is known about how students use the technology and what understandings they develop when it is used to learn new mathematical concepts".

Various works [Guv12] studied teaching math with technology; including not only computers, but also personal hand-held calculators. Guven *et al.* [Guv12] focused on 2D transformations performed and the paper documented a substantial improvement of students' understanding before teaching with DGS Cabri. However, the authors concluded that further research of the depth of understanding is necessary. Although various programs, applications, and approaches have been utilized, we argue that using specialized applications that employ advanced concepts from Computer Graphics and real-time interactive visualization may lead to better learning outcomes – that is also the goal of our paper.

One of the areas that requires strong understanding of geometric transformations is **robotics**, in particular, forward and inverse kinematics, which describe the movement of the articulated robotic arms. Application toolboxes exist for dealing with robot modeling and planning, such as Robot Computer Aided Analysis and Design (RCAAD) [DBBC*99] or Rapid Analysis Manipulator Program (RAMP) [HT96], but these tools are not designed with educational purposes in mind or rely on commercial software.An approach used the Unity3D game engine and Python [AHAGA17]. However, students had trouble with complexity of Unity3D, as the system does not focus on inverse kinematics, and it also only uses a simple GUI. We argue that fundamental geometric transformations that are in the core of forward and inverse kinematics can be well-understood by using real-time interactive concepts that will show the mathematics and the 3D transformation simultaneously.

Computer Graphics Applications: Visualizations have been considered significant to understanding transformations as early as 1976 [MD76]. They are also the cornerstone of Computer Graphics and they have been used for teaching since [Nai96]. The learning modules were implemented as applications in C/C++ or as Java applets [LS01] and they use graphical libraries such as OpenGL, Java3D, or Processing. There are many examples of code that accompany computer graphics textbooks, web sites, general books focusing on OpenGL and Direct3D, etc. However, they usually focus on one problem, or they embed models, commands and their parameters into the code, leaving little space to modify it.

There are higher level approaches that prepare specialized demos for explaining a particular concept by allowing the user to modify the parameters of a function and see the effect on the scene. An example is the model and projection transformation by

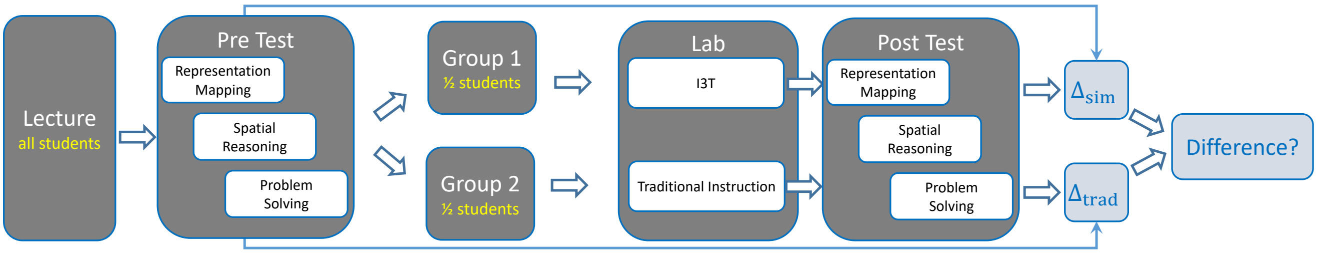

Figure 2: *Method overview (dark boxes are actions, light ones are data) starts with the pretest that measures the representational fluency (initial level of subject knowledge) of the students. The participants are then divided into two random groups and one is exposed to the traditional instruction, whereas the other uses the simulation system. Their representational fluency is measured in a post test and the gain (Δ between the pretest and post test) is quantified. Our hypothesis is that the gain from simulation is higher.*

Robins [Rob01] that was adapted by Ahn [Ahn13] or the framework for mental manipulation of 3D images [MKv06]. An application that attempts to explain the concept of forward kinematics by using interactive application is the Robot Arm [CSE03]. These systems are close to our approach, because they are interactive and allow for direct visualization of the transformation parameters. However, these approaches allow users to modify the transformation parameters in a predefined setup only and they often specialize in a limited task. Programmable editor by Andújuar [AV06] is a Qt-based implementation of examples by [Rob01] enriched by a script editor for setting the parameters and by an export of the OpenGL commands into a C++ code. We overcame the limitations of the previous approaches by presenting a modular system for teaching transformations. Our I3T system allows for displaying and modification of transformation matrices, positioning of the transformations in the transformation hierarchy, re-connection using graphical environment, and uses modern OpenGL. Moreover, contrary to the previous work, we present an evaluation of the usage of the system and its benefit for teaching transformations.

3. Theoretical Framework

The theoretical framework that guided our investigation relates to representational competence and spatial reasoning. Research on graphical representations in education has been investigated in areas of mathematics [Pap80], physics [She01, JCW08], chemistry [KCRM00], biology [Rot03, BRM99], and engineering [BL10, CMLC11, Bli10], among others. However, this research has shown no effective ways of using graphical representations meaningfully [And99], and overall, student representational proficiencies have shown unsatisfactory results [And99]. For instance, instead of treating representations as thinking tools, students tended to treat them as school tasks [And99]. Also, students struggle in applying or mapping knowledge about graphical representations while simultaneously comprehending new domain knowledge [BL10, Kin94, MSS*17].

Representational competence refers to the ability to use, manipulate, and communicate with representations [KR05]. Its particular form that students need to develop consists of the ability not only to produce, read, manipulate, interpret, and reinterpret representations [SSB93], but also to comprehend equivalences in different modes of expression and to learn, transform, and apply information from one representation to another [BL10]. Representational competence of geometric transformations specifically, requires students to understand the relationship between multiple forms of representations: diagrams, graphs representing geometries in a coordinate system, mathematical equations, and matrices.

Understanding geometric transformations requires students to also develop spatial thinking, and to be able to perform operations on such representations such as: translation, scaling, rotation, and shear. Spatial thinking involves three components: "concepts of space, tools of representation, and processes of reasoning" [L*07]. Together, representational competence and spatial thinking play fundamental roles throughout elementary and secondary education. However, instructional supports for the explicit teaching and learning of spatial strategies, along with affordances to help students develop representational competence, are currently lacking.

4. Experiment Overview

Figure 2 shows an overview of our experiment. We did not assume any particular knowledge of the topic of the participants, but they were required to be familiar with general courses of algebra that usually do not put particular stress on interpretation of the algebraic concepts. We also measured the subjects' representational fluency by a pretest.

The students were divided into two equal but random groups and no attempt to balance the abilities of the two groups has been made. One group was exposed to the traditional instruction and the second group used our application I3T.

After the treatment, the same post test was applied to both group and the representational fluency was evaluated and compared to the input. The actual gain of the I3T simulation treatment and traditional instruction was quantified as Δ_{I3T} and Δ_{trad} respectively.

Our hypothesis was that the gain from the simulation is higher than from the traditional instruction, e.g., $\mathcal{H} : \Delta_{I3T} \gg \Delta_{trad}$. Below we describe details of each individual step.

5. Methods

Based on the study goals and the topics, we grouped the treatment and the questions into four sections: 1) basic concepts, 2) matrix representation of transformations, 3) transformation matrices, and

4) composition of transformations including the order of transformations and transform inverse.

The *1) basic concepts* section was not part of the treatment, as these concepts were assumed to be known from linear algebra courses. The participants were required to have a fundamental knowledge of linear algebra including matrix multiplication, transformations of points by matrices, 3D modeling, and construction of 3D objects. These are common requirements for graduate students that form our testing groups.

The *2) matrix representation* section introduced a 3×3 linear and a 4×4 affine transformation matrix, altogether with the concept of homogeneous coordinates of a point and vector.

The *3) individual transformation matrix* section provided examples of members of both groups, such as uniform and non-uniform scaling, different variants of rotation, and translation.

The *4) composition & inverse* section discussed the order of transformations and a translation-rotation TR matrix decomposition, the examples of directly written inverse matrices for simple transformations followed by matrix inverse. Next, the concept of rigid transformation and the rigidness test were described. Finally, Gimbal lock (see Figure 4) and rotation around a given coordinate frame were explained, this allows an in-depth understanding of advanced concepts.

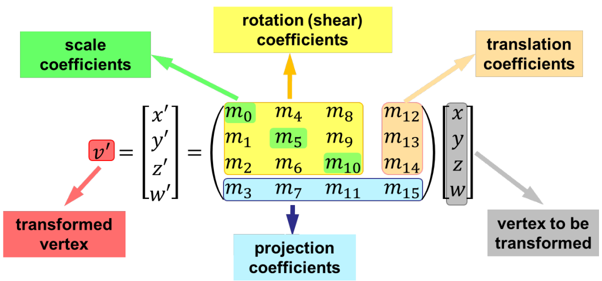

Figure 3: *An affine and a projection transformation matrix.*

There is an assumption that participants vary in performing 3D transformation and this can be quantified by the Purdue Spatial Reasoning Test [Gua80]. We have performed this testing on all participants and we show how different spatial abilities are reflected in comprehension of geometric transformations in Section 6.

All groups were discussed in both variants of the treatment; a traditional method and by means of the I3T tool.

5.1. Traditional Approach

Traditional treatment method used the approach that is commonly followed in algebra and computer graphics textbooks; i.e., text, combined with diagrams and examples of matrices. The order of explanation of the subject followed the order of sections defined in the previous section.

We started with the description of homogeneous coordinates followed by the linear and affine transformation represented by matrices. We then explained the meaning of individual values in their rows and columns (Figure 3). Examples of linear and affine transformation matrices were followed by a longer text describing more complex topics, such as inverse of simple transformations, concept of rigidness, and the Gimbal lock (Figure 4). Finally, the explanation returned to the fundamentals; i.e., the construction matrix for changing the bases and rotation around a given point.

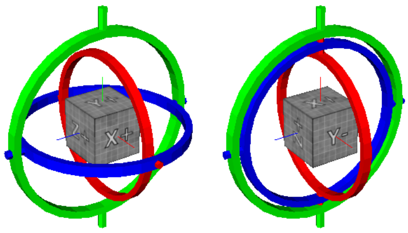

Figure 4: *Three yzx Gimbals in the initial position (left) and the Gimbal lock: a lost of one degree of freedom around the green y-axis (right).*

5.2. I3T

The supporting material for the I3T treatment differed from the traditional group by including an additional introductory chapter describing the user interface, commands, including the menu structure and by adding the snapshots of the I3T tool usage.

The I3T tool consists of a window split into: a *workspace*, where the user constructs the scene graph (Figure 1 (left)) and the *3D scene* with an immediate 3D representation of the structure generated by the graph (Figure 1 (right) and the accompanying video).

The *workspace* is an interactive environment where the user places 3D objects and transformation matrices and connects them into a scene graph. The basic building block is a small rectangle with the title and contents, called *box*. The matrices are yellow boxes with a 4×4 grid of real numbers. Matrices affect the object by being combined in the scene graph as they are put into a larger box called a *sequence*.

Sequences have inputs and outputs marked by small icons. The [x] icon represents matrix multiplication and matrices are multiplied by using the left-to-right order. The remaining icons represent either input or output of the copy of the matrix ▲▼**M**.

The key benefit of the I3T tool is its interactivity. The user can construct the scene graph interactively by ordering the matrices into sequences, the sequences into graphs, and also interactively modify the values in the matrices. The effect of the transformation is immediately shown in the 3D scene view (Figure 1 and the accompanying video).

The learners who used the I3T were provided prepared examples of some scenes. They could modify the values in matrices or change the order of the matrices.

5.3. Spatial Abilities

In our experiment we used the Purdue University Spatial Reasoning Test [Gua80] which consists of a sequence of images with increasing difficulty, where the participant is asked to visualize rotations in 3D. Participants were shown a complex geometrical object in a 3D view and the same object from a different angle. At the same time, they were shown another object that had five different counterparts. The task was to identify which of the five objects represented the same rotation as the first pair. An example in Figure 5 from [Gua80] shows a question from the spatial reasoning test we used.

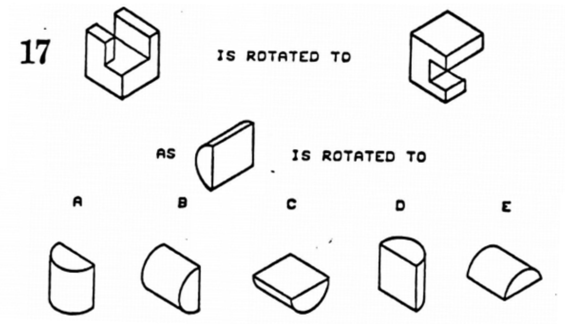

Figure 5: *An example question from the Purdue spatial test [Gua80] used during the pretest.*

5.4. Procedures

The experiment was performed at a Midwest university in the USA as a part of a graduate course of *Introduction to Computer Graphics Programming* with OpenGL. The participants have already taken the basics of projections and camera transformations, but these tasks were not a part of the testing.

Eighteen **participants** (7 F, 11 M; 10 BS, 6 MS, and 2 Ph.D. students) were recruited from the computer graphics class and their ages ranged from 20 to 36 years-old ($mean = 25$, $SD = 4.2$).

The used **computer hardware** consisted of a desktop PC computer with LED display, keyboard, and optical mouse with three buttons. All computers were equipped with the same hardware: each computer had an Intel Xeon CPU clocked at 3 GHz with 16GB of memory and NVIDIA GeForce 970 Graphics Card.

We explained the testing **procedure** and its objective, i.e., measuring the difference of learning transformation by using different learning methods. The participants were informed that the testing was anonymous and no personally identifiable data was collected.

Years of experience in Computer Graphics, education level, gender, and age were the demographic data collected from the participants. This took about five minutes.

The **pretest phase** consisted of three steps: representation mapping, problem solving, and spatial reasoning. *Representation mapping* was a multiple choice test with One-Best-Answer question test taking 10 minutes. The *problem solving* required participants to write an open response to each question and also taking 10 minutes. At the end of the pretest phase, the researchers applied the *Purdue Spatial Visualization Test* [Gua80] that measured participants' spatial ability (see Section 5.3). The spatial reasoning test took 15 minutes.

After the pretest was completed, the participants were randomly split into two halves and we started the **treatment phase** which took 30 minutes. One group used the I3T tool, while the second group used the printed teaching materials. The I3T included step-by-step visual modifications of transformations with visual feedback (see Section 5.2).

The **post test phase** consisted of the same first two steps as the pretest, but it did not include the spatial reasoning test, because spatial reasoning would very unlikely change during such a short period of time and the treatment would not have had a significant affect on that either.

At the end of the testing, the participants were asked for their subjective judgment about the material and they were also asked to provide written comments.

6. Evaluation and Results

Our objective was to verify the hypothesis that learners would perform better (faster and deeper understanding) by using the I3T than by using the traditional learning approach. Our evaluation had three steps: First, we measured the improvement of the traditional group (post test minus pretest). Second, we measured the improvement of the I3T group. Third, we compared the difference of improvements between the two groups. Below, we describe an in-depth analysis and we also discuss evaluation per question groups.

Based on the comparison of pretest and post test, we have classified each answer of each participant of the first test into one of the following four categories: The answer:

1. *improved* (wrong in pretest → correct in post test),
2. *stayed correct* (correct in pretest → correct in post test),
3. *stayed wrong* (wrong in pretest → wrong in post test), or
4. *worsened* (correct in pretest → wrong in post test).

We also noticed that the effect of spatial reasoning on the performance of learning transformations was negligible (Section 6.5). Tables 1 and 2 summarize the findings that are further analyzed below.

	Improved	Stayed correct	Stayed wrong	Worsened
Traditional	**10%**	48%	**39%**	3%
I3T	**20%**	48%	**27%**	6%

Table 1: *Improvement of individual questions in Test 1.*

		Test 1			Test 2		
		pretest	posttest	delta	pretest	posttest	delta
Traditional	MEAN	5.1	5.8	0.7	3.3	4.4	0.4
	STDEV	2.5	2.6	1.0	1.8	2.6	0.9
I3T	MEAN	5.3	6.8	1.4	3.8	3.6	-0.2
	STDEV	1.8	2.2	1.7	2.6	2.5	1.2

Table 2: *The mean and the standard deviation of the results of tests 1 and 2.*

6.1. Traditional Group

The traditional group showed improvement in the **representation mapping questionnaire (test 1)** and the overall improvement (number of correct answers) of the traditional group was statistically significant. The participants in average improved in 0.78 questions and worsened in 0.11 of 10 questions, resulting in per test improvement from ($mean = 5.1, SD = 2.5$) to ($mean = 5.8, SD = 2.6$), with a p-value=0.0403.

Based on individual questions, for the participants in the traditional treatment group, the answers to test 1 in **10% of questions** *improved*, in 48% of questions *stayed correct*, in 39% of questions *stayed wrong*, and in 3% of questions *worsened*.

The overall improvement (number of correct answers) for the traditional group in **problem solving (test 2)** was not statistically significant, regardless of the fact that the participants in average improved from ($mean = 3.3, SD = 1.8$) to ($mean = 4.4, SD = 2.6$) of maximum 10 points per test with a p-value=0.145. The probable cause was that two participants opt out of the second test and one participant worsened.

6.2. I3T Group

The I3T group also improved in the **representation mapping questionnaire (test 1)** and the overall improvement (number of correct answers) was also statistically significant. The participants in average improved in 1.56 questions (double to the traditional group) and worsened in 0.11 of 10 questions, resulting in per test improvement from ($mean = 5.3, SD = 1.8$) to ($mean = 6.8, SD = 2.2$), with a p-value=0.016.

Based on individual questions of the participants in I3T treatment group the answers to test 1 in **20% of questions** *improved* (double of the traditional group), in 48% of questions *stayed correct*, in 27% of questions *stayed wrong*, and in 6% of questions *worsened*.

Problem solving (test 2) The overall improvement (number of correct answers) for I3T group in **problem solving (test 2)** was none. The participants on average changed from ($mean = 3.8, SD = 2.6$) to ($mean = 3.6, SD = 2.5$) of maximum 10 points per test with a p-value=0.297.

6.3. I3T versus Traditional

We compared the results of the I3T group against the traditional one (see Figure 2). In the **representation mapping questionnaire (test 1)**, we compared the overall improvement from I3T group (Section 6.2) against the traditional group (Section 6.1). The performance in test 1 was higher for the I3T group ($mean = 1.4, SD = 1.7$) as compared to the performance in the Traditional group ($mean = 0.7, SD = 1.0$). However, based on the results of the paired t-test the difference was not significant (p-value=0.393).

Based on the individual questions, the participants in I3T treatment group improved in two times more questions (18 vs. 9, that is 20% vs. 10%) and worsened in 3% more questions (5 vs. 3). Four participants in the I3T group improved in one question, four in two questions, and one improved in five questions. In the traditional group, three participants improved in one question, and two in two questions. These results suggest that the I3T group performed better in general on most questions.

The overall improvement (number of correct answers) for both groups in the **problem solving (test 2)** was slightly worse for the I3T group ($mean = -0.3, SD = 1.2$) to ($mean = 0.4, SD = 0.9$) as compared to the traditional group, however, this difference was not significant (p-value=0.441).

The participants from the traditional group failed in question one that dealt with the Cartesian bases.

6.4. Per Question Blocks

The traditional and I3T groups were similarly capable: 46 vs. 48 points from the first application of pretest and 30 vs. 34 points from the second one. After the treatment, both remained equal for post test 2 (31 vs. 32 points), but the I3T performed better in post test 1 (61 to 52).

Representation Mapping: Ten questions in test 1 were taken from four principal blocks:

1. **Definitions of bases (1 question):**
   I3T performed better. The traditional group worsened in 11% and did not improved, I3T improved in 22% and not worsened.
2. **Matrix representation of linear and affine transformations (4 questions):**
   - I3T performed slightly better in question dealing with determinant of a rigid transformation matrix (the traditional group improved in 22%, the I3T improved in 44% and worsened in 11%.) and in question about rotation matrix properties (11% worsened in traditional group versus zero score in I3T group, where 22% improved and 22% worsened).
   - The I3T group performed worse in one question not directly addressed by the tool: the size of 3x3 linear matrices, as the matrices in I3T are all 4x4. The traditional group improved in 33% and the I3T in 11% only.
   - The fourth question about translation matrix shown no difference, as nearly all participants knew the correct answer already in the pretest. 11% improved in both groups, leaving 11% wrong in the traditional group and none in the I3T group.
3. **Individual transformations separately (2 questions):**
   One question (identity matrix) knew nearly all already in the pretest (90% in both groups), one answer improved (homogeneous coordinates of a point), where 22% improved in traditional group and 33% improved in the I3T group.
4. **Composition and inverse (3 questions):**
   The I3T group performed well and improved in two questions dealing with composition of transformations (22% improvement in I3T group to 11% worsening in traditional group). The I3T group was slightly worse in inverse of a RT matrix, where both groups improved 11% but I3T also worsen in 22%.

Problem Solving Test 2 contained four blocks:

1. **Definition of bases vectors (3 points):**
   Similar questions as in block 1 of test 1. They were not described in the materials as they are prerequisite from algebra classes. The traditional group performed worse in 29%, I3T was worse in 29% and gained also 29% to the final score of zero.
2. **Coefficients in affine matrix (1 point):**
   This topic was equably described in both materials. The Traditional group got 14% and lost 14% resulting in score of 0, I3T lost 33%.

3. **Writing a rotation matrix(2 points):**
   Traditional group gained 29%, I3T received 22% and lost 11%.
4. **Symmetry to a given point (4 points):**
   Writing a sequence of transformations describing symmetry around a point. This question was better solved by the traditional group - got 57% and lost 14%. The I3T got and lost 11%.

6.5. Spatial reasoning

Our results from the modified version of the *Purdue spatial visualization test* PSVT showed the mean value of 19.9 and SD=5.3. This is less than mean value 23.3 and SD 5.14 reached by the control group by [Bra00], consisting of 139 individuals and our students seem to have lower spatial reasoning abilities than members of the Branoff's group.

We did not find the correlation between the Spatial reasoning test and the tests performed in our study. Overall results suggest that students increased their understanding of transformations between the pretest and post test $t = -3.04256, p = 0.00253$). The mean score in the post test was higher for I3T Group, but it was not significantly higher than the Traditional Group. Both methods were effective in helping students learn about transformations.

6.6. Written Responses

We received several positive responses such as: "I have been dreaming about an application like this, since I started dealing with transformations", or "The materials given to the treatment are helpful and I am glad I got it.".

Negative responses included "I was group 2 [traditional group] and I cannot learn well from the reading", "The wording of the problems were kind of confusing, the program shows how each matrix work (sic) but does not explain how it is calculated", "However, going through it [written material] was boring".

7. Discussion and Implications for Teaching and Learning

Our results suggest that students equally benefited from both approaches, regardless their spatial abilities. This result is relevant because previous work [SCVD13] has identified that spatial abilities are a gatekeeper for learning advanced concepts in science, engineering, and mathematics. Furthermore, previous research [SB03] has identified evidence of the relationship between students with higher visual-spatial skills performing significantly better on mathematical problem solving. Our results, however, cannot be generalized to the larger population because of the small sample size.

Students in the I3T group performed better than students in the Traditional group on questions associated with representation mapping. Previous research [SB03] has shown that graphical representations affect the structure of mental models and it has also suggested that while appropriate graphics may support learning, task-inappropriate graphics may interfere with mental model construction. Therefore, it is also needed to consider the conditions and instructional events that can result in better learning. For instance, an important aspect is the role of the teacher in the process of using technology in the classroom and the class of problems the teacher approaches. Training of teachers, mentoring during the first steps, and collegial support together with development of successful technology assisted teaching practices is necessary for successful integration of technology in teaching [HHKL08].

A limitation of the I3T is the learning curve necessary to fully comprehend it. While we attempted to create a compact and easy-to-use interface, we suspect that students may not have fully understood how the interface worked. Our system now attempts to encompass all possible cases, but it could be possible to create a simpler version that would focus solely on a subset of problems. Another limitation of our experiment could have been the its duration. While we attempted for a self-paced manner, it is not clear that all students had enough time to fully understand the problem. Another weakness is the small sample size, as well as the unknown quantity of the students/respondent's familiarity with matrices and their operations, such as students who had more or less experience/education with linear algebra.

8. Conclusions

We have introduced I3T, an interactive tool for teaching 3D geometric transformations. Our results suggest that although students in the I3T group performed higher in the graphical representation questionnaire, both groups equally benefited from their own approaches to learning, regardless their spatial reasoning abilities.

There are several potential avenues for **future work**. We could measure if the provided time for using I3T is sufficient and if any further exposure to I3T would improve the results. Also, the pretest 1 and post test 1 could switch variant *a/b* to randomize the questions. We could also inform the participants about testing a week before the test is done and we could improve the descriptive materials by adding more images demonstrating how matrix computations visually operate in the traditional lecture. An interesting question is the actual appropriateness of the test questions in relation to the interactive tool. It is possible that the interaction is not necessary to answer some of the questions while it may help others.

Acknowledgments: This work has been supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS16/237/OHK3/3T/13 (Research of Modern Computer Graphics Methods). We would also like thank to Tim McGraw for providing the time in his OpenGL course.

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